ON THE SYMMETRICAL FORM OF THE DIFFERENTIAL EQUATIONS OF PLANETARY MOTIONS.

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The differential equations usually employed in determining the motions of the planets,

$$\begin{aligned} \frac{d^2x_i}{dt^2} + \frac{k^2(1+m_i)}{r_i^3}\, x_i &= \frac{\partial R_i}{\partial x_i}, \\ \frac{d^2y_i}{dt^2} + \frac{k^2(1+m_i)}{r_i^3}\, y_i &= \frac{\partial R_i}{\partial y_i}, \qquad i = 1, 2, \ldots, n. \qquad (1) \\ \frac{d^2z_i}{dt^2} + \frac{k^2(1+m_i)}{r_i^3}\, z_i &= \frac{\partial R_i}{\partial z_i}, \end{aligned}$$

are unsymmetrical, since the functions $R_1, R_2, \ldots, R_n$ are not the same for each planet. The symmetrical form* may be obtained in the following manner :—

1. Let ξ_i, η_i, ζ_i be the coordinates of the different masses of the system M_i referred to fixed rectangular axes. Let G_i be the center of gravity of the masses $M_1, M_2, \ldots, M_i$; let x_i, y_i, z_i be the coordinates of M_i referred to three axes parallel to the fixed axes, but passing through G_{i-1}; let X_i, Y_i, Z_i, be the coordinates of G_i; $\mu_i = \sum\limits_1^i{}_\sigma m_\sigma$. If the number of the bodies be n, G_n will be the center of gravity of the system. We shall also have

$$\xi_i = X_{i-1} + x_i, \qquad (2)$$

in which $X_1 = \xi_1$; and, if we put $x_1 = 0$, we have $X_0 = \xi_1$, in which X_0 is introduced merely in order that the nomenclature in (2) may be applicable throughout.

We have, also, in accordance with the properties of centers of gravity,

$$\mu_{i-1}X_{i-1} + m_i\xi_i = \mu_i X_i; \qquad (3)$$

whence, substituting for ξ_i its value as given by (2), we obtain

$$\mu_i X_{i-1} + m_i x_i = \mu_i X_i,$$

or

$$m_i x_i = \mu_i (X_i - X_{i-1}). \qquad (4)$$

* See Tisserand's Mécanique Céleste, t. i, chap. iv, the substance of which is derived from an interesting memoir by M. R. Radau, entitled "Sur une transformation des équations differentielles de la Dynamique" (Annales de l'École Normale, 1re série, t. v).

Squaring (2) and multiplying by m_i, we have

$$m_i\xi_i^2 = m_i x_i^2 + 2m_i x_i X_{i-1} + m_i X_{i-1}^2 ;$$

whence, adding $m_i x_i (X_i - X_{i-1}) - \frac{m_i^2}{\mu_i} x_i^2 = 0$,

$$m_i\xi_i^2 = m_i \frac{\mu_{i-1}}{\mu_i} x_i^2 + m_i x_i (X_i + X_{i-1}) + m_i X_{i-1}^2 .$$

If $\mu_i (X_i - X_{i-1})$ be substituted for $m_i x_i$ in the middle term, this becomes

$$m_i\xi_i^2 = m_i \frac{\mu_{i-1}}{\mu_i} x_i^2 + \mu_i (X_i^2 - X_{i-1}^2) + m_i X_{i-1}^2$$

$$= m_i \frac{\mu_{i-1}}{\mu_i} x_i^2 + \mu_i X_i^2 - \mu_{i-1} X_{i-1}^2 ;$$

or, since $\mu_0 = 0$,

$$\sum_1^n {}_i\, m_i\xi_i^2 = \sum_1^n {}_i\, m_i \frac{\mu_{i-1}}{\mu_i} x_i^2 + \mu_n X_n^2 . \tag{5}$$

2. Equation (5) may be obtained in a still simpler manner. Equation (4) gives

$$X_i = X_{i-1} + \frac{m_i}{\mu_i} x_i ;$$

whence, substituting in (2), we have

$$\xi_i = X_i + \frac{\mu_{i-1}}{\mu_i} x_i . \tag{6}$$

From (3) we have, also,

$$m_i\xi_i = \mu_i X_i - \mu_{i-1} X_{i-1} . \tag{7}$$

Multiplying the left hand member by ξ_i, and the terms of the right hand member by the values of ξ_i given by (6) and (2), respectively, equation (7) becomes

$$m_i\xi_i^2 = \mu_i X_i^2 - \mu_{i-1} X_{i-1}^2 + \mu_{i-1} x_i (X_i - X_{i-1})$$

$$= \mu_i X_i^2 - \mu_{i-1} X_{i-1}^2 + m_i \frac{\mu_{i-1}}{\mu_i} x_i^2 ;$$

whence, as before,

$$\sum_1^n {}_i\, m_i\xi_i^2 = \sum_1^n {}_i\, m_i \frac{\mu_{i-1}}{\mu_i} x_i^2 + \mu_n X_n^2 . \tag{5}$$

3. If we differentiate (2) and (3) with reference to t, we see at once that the relations between the differentials are exactly the same as the relations

between the corresponding variables ; hence we may substitute the differentials for the variables in (5), and obtain

$$\sum_1^n{}_i\, m_i \left[\frac{d\xi_i}{dt}\right]^2 = \sum_1^n{}_i\, m_i \frac{\mu_{i-1}}{\mu_i}\left[\frac{dx_i}{dt}\right]^2 + \mu_n \left[\frac{dX_n}{dt}\right]^2. \tag{8}$$

There are also relations similar to (5) and (8) for the coordinates η and ζ. Adding, we have

$$\sum_1^n{}_i\, m_i \rho_i^2 = \sum_1^n{}_i\, m_i \frac{\mu_{i-1}}{\mu_i} r_i^2 + \mu_n R_n^2, \tag{9}$$

in which $\rho_i^2 = \xi_i^2 + \eta_i^2 + \zeta_i^2$, $r_i^2 = x_i^2 + y_i^2 + z_i^2$, and $R_n^2 = X_n^2 + Y_n^2 + Z_n^2$; also

$$\begin{aligned} 2T \equiv & \sum_1^n{}_i\, m_i \left[\left[\frac{d\xi_i}{dt}\right]^2 + \left[\frac{d\eta_i}{dt}\right]^2 + \left[\frac{d\zeta_i}{dt}\right]^2\right] \\ = & \sum_1^n{}_i\, m_i \frac{\mu_{i-1}}{\mu_i}\left[\left[\frac{dx_i}{dt}\right]^2 + \left[\frac{dy_i}{dt}\right]^2 + \left[\frac{dz_i}{dt}\right]^2\right] \\ & + \mu_n\left[\left[\frac{dX_n}{dt}\right]^2 + \left[\frac{dY_n}{dt}\right]^2 + \left[\frac{dZ_n}{dt}\right]^2\right]. \end{aligned} \tag{10}$$

4. An examination of (6) and (2) shows that we may write

$$\frac{d\eta_i}{dt} = \frac{dY_i}{dt} + \frac{\mu_{i-1}}{\mu_i}\frac{dy_i}{dt} = \frac{dY_{i-1}}{dt} + \frac{dy_i}{dt}. \tag{11}$$

Multiplying $m_i\xi_i$ by the first member of (11), $\mu_i X_i$ by the second, and $\mu_{i-1}X_{i-1}$ by the third, equation (7) becomes

$$\begin{aligned} m_i\xi_i \frac{d\eta_i}{dt} &= \mu_i X_i \frac{dY_i}{dt} + \mu_{i-1}X_i \frac{dy_i}{dt} - \mu_{i-1}X_{i-1}\frac{dY_{i-1}}{dt} - \mu_{i-1}X_{i-1}\frac{dy_i}{dt} \\ &= \mu_i X_i \frac{dY_i}{dt} - \mu_{i-1}X_{i-1}\frac{dY_{i-1}}{dt} + m_i \frac{\mu_{i-1}}{\mu_i} x_i \frac{dy_i}{dt}. \end{aligned}$$

A similar expression can be readily obtained for $m_i\eta_i \dfrac{d\xi_i}{dt}$; whence

$$\begin{aligned} \sum_1^n{}_i m_i \left[\xi_i \frac{d\eta_i}{dt} - \eta_i \frac{d\xi_i}{dt}\right] = \mu_n \left[X_n \frac{dY_n}{dt} - Y_n \frac{dX_n}{dt}\right] & \\ + \sum_1^n{}_i m_i \frac{\mu_{i-1}}{\mu_i}\left[x_i \frac{dy_i}{dt} - y_i \frac{dx_i}{dt}\right]. & \end{aligned} \tag{12}$$

5. If U be the force function of the system, we may put

$$P = \frac{\partial T}{\partial \frac{dX_n}{dt}}, \qquad P_1 = \frac{\partial T}{\partial \frac{dY}{dt}}, \qquad P_2 = \frac{\partial T}{\partial \frac{dZ}{\partial t}};$$

$$p_{3i} = \frac{\partial T}{\partial \frac{dx_i}{dt}}, \qquad p_{3i+1} = \frac{\partial T}{\partial \frac{dy_i}{dt}}, \qquad p_{3i+2} = \frac{\partial T}{\partial \frac{dz_i}{dt}};$$

and write the equations of motion in the well-known canonical form,

$$\begin{aligned} \frac{dX_n}{dt} &= \frac{\partial(T-U)}{\partial P}, \ldots; \qquad \frac{dx_i}{dt} = \frac{\partial(T-U)}{\partial p_{3i}}, \ldots; \\ \frac{dP}{dt} &= -\frac{\partial(T-U)}{\partial X_n}, \ldots; \quad \frac{dp_{3i}}{dt} = -\frac{\partial(T-U)}{\partial x_i}, \ldots. \end{aligned} \tag{13}$$

Since U does not contain X_n, Y_n, Z_n, but only the differences $\xi_i - \xi_j$, $\eta_i - \eta_j$, $\zeta_i - \zeta_j$, equations (10) and (13) give

$$\frac{d^2X_n}{dt^2} = \frac{d^2Y_n}{dt^2} = \frac{d^2Z_n}{dt^2} = 0;$$

whence, by integration,

$$X_n = \alpha t + \alpha', \quad Y_n = \beta t + \beta', \quad Z_n = \gamma t + \gamma', \tag{14}$$

in which α, α', β, β', γ, γ' are arbitrary constants.

Also, since T does not contain the x_i, y_i, z_i, but only the velocities, (10) and (13) give

$$m_i \frac{\mu_{i-1}}{\mu_i} \frac{d^2x_i}{dt^2} = \frac{\partial U}{\partial x_i}, \quad m_i \frac{\mu_{i-1}}{\mu_i} \frac{d^2y_i}{dt^2} = \frac{\partial U}{\partial y_i}, \quad m_i \frac{\mu_{i-1}}{\mu_i} \frac{d^2z_i}{dt^2} = \frac{\partial U}{\partial z_i}, \tag{15}$$

which have the symmetrical form desired.